# Chapter 4

## DERIVATIVES


Imagine a person speeding down the highway reaching speeds of 140 km/h. He hears the police siren in the distance, and his rear-view mirror is suddenly full of flashing red lights. As he pulls over and the police officer tells him he was going 140 km/h, he points out that because he has travelled the 200 km from home in two hours, his average speed is within the 100 km/h limit. Nice try, but we all know that police charge speeders based on their instantaneous speed, not their average speed. Furthermore, in business, responding rapidly to instantaneous changes is the key to success; opportunities can pass you by if you wait to observe long-term or average trends. In calculus, the derivative is a tool for finding instantaneous rates of change. This chapter shows how the derivative can be determined and applied in a great variety of circumstances.

**CHAPTER EXPECTATIONS** In this chapter, you will

- determine the limit of a function, and understand limits can give information about graphs of functions, **Section 4.1**
- understand and determine derivatives of polynomial and simple rational functions from first principles, **Section 4.1**
- identify examples of functions that are not differentiable, **Section 4.1**
- identify rate of change **Section 4.2**
- identify composition as two functions applied in succession, **Section 4.5**
- understand that the composition of functions exists only when the ranges overlap, **Section 4.5**
- determine the composition of two functions expressed in notation, and decompose a given composite function into its parts, **Section 4.5**
- justify and use the rules for determining derivatives, **Section 4.2, 4.3, 4.4, 4.6**
- describe the effect of the composition of inverse functions, $(f(f^{-1}(x)) = x)$, **Section 4.5**
- determine derivatives, using implicit differentiation, **Section 4.6**
- make inferences from models of applications and compare the inferences with the original hypotheses, **Section 4.1, 4.6**

# Review of Prerequisite Skills

Before beginning your study of derivatives, it may be helpful to review the following concepts from previous courses and chapters:

- Finding the properties of exponents
- Simplifying radical expressions
- Finding the slopes of parallel and perpendicular lines
- Simplifying rational expressions
- Expanding and factoring algebraic expressions

## Exercise

1. Use the exponent laws to simplify each of the following expressions. Express your answers with positive exponents.

   a. $5^4 \times 5^7$ b. $a^5 \times a^3$ c. $(4^9)^2$

   d. $(-2a^2)^3$ e. $2m^6 \times 3m^7$ f. $\dfrac{4p^7 \times 6p^9}{12p^{15}}$

   g. $(a^4b^{-5})(a^{-6}b^{-2})$ h. $(3e^6)(2e^3)^4$ i. $\dfrac{(3a^{-4})[2a^3(-b)^3]}{12a^5b^2}$

2. Simplify and write each expression in exponential form.

   a. $\left(x^{\frac{1}{2}}\right)\left(x^{\frac{2}{3}}\right)$ b. $\left(8x^6\right)^{\frac{2}{3}}$ c. $\dfrac{\sqrt{a}\sqrt[3]{a}}{\sqrt[6]{a}}$

3. Determine the slope of a line perpendicular to a tangent that has the following slopes:

   a. $\frac{2}{3}$ b. $-\frac{1}{2}$ c. $\frac{5}{3}$ d. $-1$

4. Determine the equation of each of the following lines:

   a. passing through points $A(-3, 4)$ and $B(9, -2)$.

   b. passing through point $A(-2, 5)$ and parallel to the line with equation $3x - 2y = 5$.

   c. perpendicular to the line having the equation $y = \frac{3}{4}x - 6$, and passing through point $(4, -3)$.

5. Expand and collect like terms.

a. $(x - 3y)(2x + y)$

b. $(x - 2)(x^2 - 3x + 4)$

c. $(6x - 3)(2x + 7)$

d. $2(x + y) - 5(3x - 8y)$

e. $(2x - 3y)^2 + (5x + y)^2$

f. $3x(2x - y)^2 - x(5x - y)(5x + y)$

6. Simplify each expression.

a. $\dfrac{3x(x + 2)}{x^2} \times \dfrac{5x^3}{2x(x + 2)}$

b. $\dfrac{y}{(y + 2)(y - 5)} \times \dfrac{(y - 5)^2}{4y^3}$

c. $\dfrac{4}{(h + k)} \div \dfrac{9}{2(h + k)}$

d. $\dfrac{(x + y)(x - y)}{5(x - y)} \div \dfrac{(x + y)^3}{10}$

e. $\dfrac{x - 7}{2x} + \dfrac{5x}{x - 1}$

f. $\dfrac{x + 1}{x - 2} - \dfrac{x + 2}{x + 3}$

7. Factor completely.

a. $10a^2 - 6a$

b. $4k^2 - 9$

c. $x^2 + 4x - 32$

d. $y^2 - 11y - 42$

e. $3a^2 - 4a - 7$

f. $6y^2 + 17y + 10$

g. $x^4 - 1$

h. $x^3 - y^3$

i. $r^4 - 5r^2 + 4$

8. Use the Factor Theorem to factor the following:

a. $a^3 - b^3$

b. $a^5 - b^5$

c. $a^7 - b^7$

d. $a^n - b^n$

9. Rationalize the denominator in each of the following:

a. $\dfrac{3}{\sqrt{2}}$

b. $\dfrac{4 - \sqrt{2}}{\sqrt{3}}$

c. $\dfrac{2 + 3\sqrt{2}}{}$

d. $\dfrac{3\sqrt{2} - 4\sqrt{3}}{}$

Career Link investigate

## CHAPTER 4: THE ELASTICITY OF DEMAND

Have you ever wondered how businesses set prices for their goods and services? One of the most important ideas in marketing is the *elasticity of demand*, or the response of consumers to a change in price. Consumers respond very differently to a change in price of a staple item, such as bread, as compared to a luxury item such as jewellery. A family would probably still buy the same quantity of bread if the price increased by 20%. This is called *inelastic* demand. If the price of a gold chain, however, were increased by 20%, it is likely sales would decrease 40% or more. This is called *elastic* demand. Mathematically, elasticity is defined as the relative (percentage) change in the number demanded $\left(\frac{\Delta n}{n}\right)$ divided by the relative (percentage) change in the price $\left(\frac{\Delta p}{p}\right)$:

$$E = -\frac{\frac{\Delta n}{n}}{\frac{\Delta p}{p}}.$$

For example, if a store increased the price of a CD from \$17.99 to \$19.99 and the number sold per week went from 120 to 80, the elasticity would be

$$E = -\frac{\frac{(80 - 120)}{120}}{\frac{(19.99 - 17.99)}{17.99}} = 3.00.$$

The elasticity of 3.00 means that the change in demand is three times as large, in percentage terms, as the change in price. The CDs have an elastic demand because a small change in price can cause a large change in demand. In general, goods or services with elasticities greater than one ($E > 1$) are considered elastic (e.g., new cars), and those with elasticities less than one ($E < 1$) are inelastic (e.g., milk). In our example, we calculated the average elasticity between two price levels, but in reality, businesses want to know the elasticity at a specific or *instantaneous* price level. In this chapter, you will develop the rules of differentiation that will enable you to calculate the instantaneous rate of change for several classes of functions.

### Case Study — Marketer: Product Pricing

In addition to developing advertising strategies, marketing departments also conduct research into and make decisions on pricing. The demand–price relationship for weekly movie rentals at a convenience store is $n(p) = \frac{500}{p}$, where $n(p)$ is demand and $p$ is price.

#### DISCUSSION QUESTIONS

1. Generate two lists, each with at least five goods and services that you have purchased recently, classifying each of the goods and services as having elastic or inelastic demand.
2. Discuss how elasticity might be used in a business to make decisions about setting price levels. Give specific examples for elastic and inelastic goods.
3. Calculate and discuss the elasticity if the movie rental fee increases from \$1.99 to \$2.99.

# Section 4.1 — The Derivative Function

In this chapter, we will extend the concepts of the slope of a tangent and the rate of change to introduce the **derivative**. We will be examining the methods of differentiation, which we can use to determine the derivatives of polynomial and rational functions. These methods include the use of the Power, Sum and Difference, Product and Quotient Rules, as well as the Chain Rule for the composition of functions.

In the previous chapter, we encountered limits of the form $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$. This limit has two interpretations: the slope of the tangent to the graph $y = f(x)$ at the point $(a, f(a))$, and the rate of change of $y = f(x)$ with respect to $x$ at $x = a$. Since this limit plays a central role in calculus, it is given a name and a concise notation. The limit is called the **derivative of $f(x)$ at $x = a$**. It is denoted by $f'(a)$ and is read as "$f$ prime of $a$."

**The derivative of $f$ at the number $a$ is given by**

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

**provided this limit exists.**

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**EXAMPLE 1** Find the derivative of $f(x) = x^2$ at $x = -3$.

**Solution**

Using the definition, the derivative at $x = -3$ is given by

$$\begin{aligned} f'(-3) &= \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} \\ &= \lim_{h \to 0} \frac{(-3+h)^2 - (-3)^2}{h} \\ &= \lim_{h \to 0} \frac{9 - 6h + h^2 - 9}{h} \\ &= \lim_{h \to 0} \frac{h(-6+h)}{h} \\ &= \lim_{h \to 0} (-6+h) \\ &= -6. \end{aligned}$$

Therefore, the derivative of $f(x) = x^2$ at $x = -3$ is $-6$.
An alternative way of writing the derivative of $f$ at the number $a$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

In applications where we are required to find the value of the derivative for a number of particular values of $x$, using the definition repeatedly for each value is tedious.

The next example illustrates the efficiency of calculating the derivative of $f(x)$ at an arbitrary value of $x$ and using the result to determine the derivatives at a number of particular $x$-values.

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**EXAMPLE 2**

a. Find the derivative of $f(x) = x^2$ at an arbitrary value of $x$.

b. Determine the slopes of the tangents to the parabola $y = x^2$ at $x = -2$, 0, and 1.

**Solution**

a. Using the definition,
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x. \end{aligned}$$

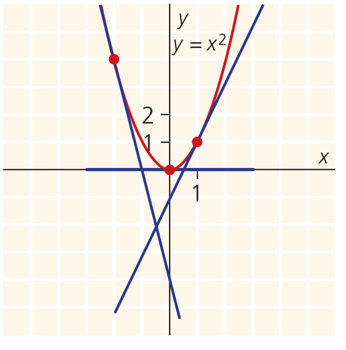
The derivative of $f(x) = x^2$ at an arbitrary value of $x$ is $f'(x) = 2x$.

b. The required slopes of the tangents to $y = x^2$ are obtained by evaluating the derivative $f'(x) = 2x$ at the given $x$-values. We obtain the slopes by substituting for $x$:

$$\begin{aligned} f'(-2) &= -4 \\ f'(0) &= 0 \\ f'(1) &= 2. \end{aligned}$$

The slopes are $-4$, 0, and 2, respectively.

The tangents to the parabola $y = x^2$ at $x = -2$, 0, and 1 are shown.



**INVESTIGATION**

1. Find the derivative with respect to $x$ of each of the following functions:
   i) $f(x) = x^3$ ii) $f(x) = x^4$ iii) $f(x) = x$
2. In Example 2, we showed that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. Referring to step 1, what pattern do you see developing?
3. Use the pattern of step 2 to predict the derivative of $f(x) = x^{39}$.
4. What do you think $f'(x)$ would be for $f(x) = x^n$, where $n$ is a positive integer?

## The Derivative Function

The derivative of $f$ at $x = a$ is a number $f'(a)$. If we let $a$ be arbitrary and assume a general value in the domain of $f$, the derivative $f'$ is a function. For example, if $f(x) = x^2$, $f'(x) = 2x$, which is itself a function.

**The derivative of $f(x)$ with respect to $x$ is the function $f'(x)$ where**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

**provided this limit exists.**

The limit $f'(x)$ is read "$f$ prime of $x$." This notation was developed by Joseph Louis Lagrange (1736–1813), a French mathematician.

In Chapter 3, we discussed velocity at a point. We can now define (instantaneous) velocity as the derivative of position with respect to time. If a body's position at time $t$ is $s(t)$, then the body's velocity at time $t$ is

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}.$$

Likewise, the (instantaneous) rate of change of $f(x)$ with respect to $x$ is the function $f'(x)$, whose value is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

**EXAMPLE 3**

Find the derivative $f'(t)$ of the function $f(t) = \sqrt{t}$, $t \geq 0$.

### Solution

Using the definition,

$$\begin{aligned} f'(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h}\left(\frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + \sqrt{t}}\right) && \text{(Rationalizing the numerator)} \\ &= \lim_{h \to 0} \frac{(t+h) - t}{h(\sqrt{t+h} + \sqrt{t})} \end{aligned}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{t+h} + \sqrt{t})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{t+h} + \sqrt{t}}$$

$$= \frac{1}{2\sqrt{t}}, \text{ for } t > 0.$$

Note that $f(t) = \sqrt{t}$ is defined for all $t \geq 0$, whereas its derivative $f'(t) = \frac{1}{2\sqrt{t}}$ is defined for only $t > 0$. From this, we can see that a function need not have a derivative throughout its entire domain.

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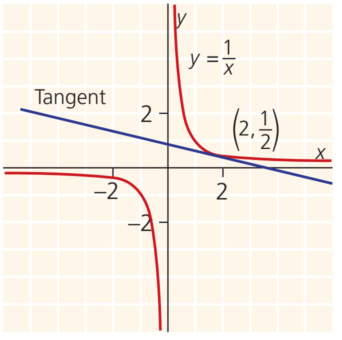
**EXAMPLE 4** Find an equation of the tangent to the graph of $f(x) = \frac{1}{x}$ at the point where $x = 2$.

### Solution

When $x = 2$, $y = \frac{1}{2}$. The graph of $y = \frac{1}{x}$, the point $\left(2, \frac{1}{2}\right)$, and the tangent at the point are shown. First find $f'(x)$.




$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \qquad \left(f(x) = \frac{1}{x};\ f(x+h) = \frac{1}{x+h}\right)$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x} \qquad \text{(Simplify the fraction)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h)x}$$

$$= -\frac{1}{x^2}$$

The slope of the tangent at $x = 2$ is $m = f'(2) = -\frac{1}{4}$. The equation of the tangent is $y - \frac{1}{2} = -\frac{1}{4}(x - 2)$, or in standard form, $x + 4y - 4 = 0$ .

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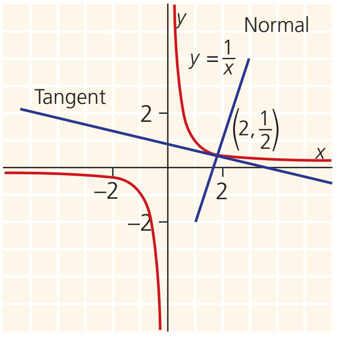
**EXAMPLE 5** Find an equation of the line that is perpendicular to the tangent to the graph of $f(x) = \frac{1}{x}$ at $x = 2$ and that intersects it at the point of tangency.

### Solution

From Example 4, we found that the slope of the tangent is $f'(2) = -\frac{1}{4}$ and the point of tangency is $\left(2, \frac{1}{2}\right)$. The perpendicular line has slope 4, the negative reciprocal of $-\frac{1}{4}$. Therefore, the required equation is $y - \frac{1}{2} = 4(x - 2)$ or $8x - 2y - 15 = 0$.

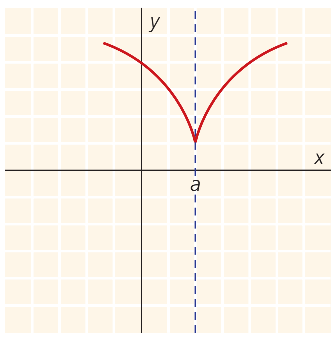


The line we found in Example 5 has a name.

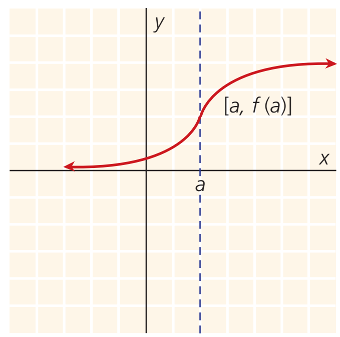
**The normal to the graph of $f$ at point $P$ is the line that is perpendicular to the tangent at $P$.**

## The Existence of Derivatives

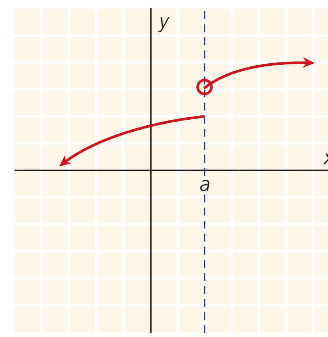
A function $f$ is said to be **differentiable** at $a$ if $f'(a)$ exists. At points where $f$ is not differentiable, we say that the *derivative does not exist*. Three common ways for a derivative to fail to exist are shown.




Cusp




Vertical Tangent




Discontinuity

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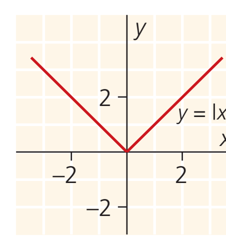
**EXAMPLE 6** Show that the absolute value function $f(x) = |x|$ is not differentiable at $x = 0$.

### Solution




The graph of $f(x) = |x|$ is shown. Because the slope for $x < 0$ is $-1$ while the slope for $x > 0$ is $+1$, the graph has a "corner" at $(0, 0)$, which prevents a unique tangent being drawn there. We can show this using the definition of a derivative.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h) - 0}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h} \end{aligned}$$

Now, we will consider one-sided limits.
$|h| = h$ when $h > 0$ and $|h| = -h$ when $h < 0$.

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} (1) = 1$$

Since the left-hand and right-hand limits are not the same, the derivative does not exist.

From Example 6, we conclude that it is possible for a function to be **continuous** at a point and yet to be *not differentiable* at that point.

## Other Notation for Derivatives

Symbols other than $f'(x)$ are often used to denote the derivative. If $y = f(x)$, the symbols $y'$ and $\frac{dy}{dx}$ are used instead of $f'(x)$. The notation $\frac{dy}{dx}$ was originally used by Leibniz and is read "dee $y$ by dee $x$." For example, if $y = x^2$, the derivative is $y' = 2x$ or, in Leibniz notation, $\frac{dy}{dx} = 2x$. Similarly, in Example 4, we showed that if $y = \frac{1}{x}$, then $\frac{dy}{dx} = \frac{1}{x^2}$. The Leibniz notation reminds us of the process by which the derivative is obtained — namely, as the limit of a difference quotient:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

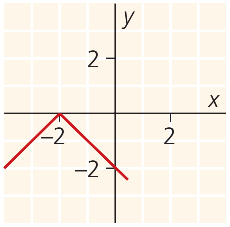
By omitting $y$ and $f$ altogether, we can combine these statements and write $\frac{d}{dx}(x^2) = 2x$, which is read "the derivative of $x^2$ with respect to $x$ is $2x$."
It is important to note that $\frac{dy}{dx}$ is *not a fraction*.

# Exercise 4.1

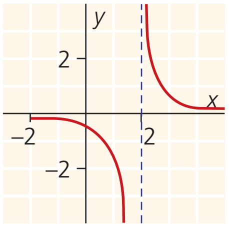
### Part A

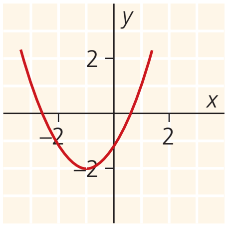
1. State the domain on which $f$ is differentiable.

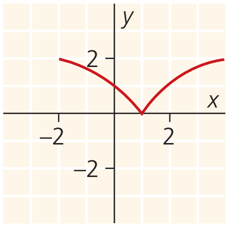
a.

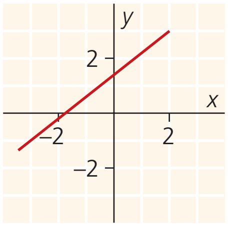


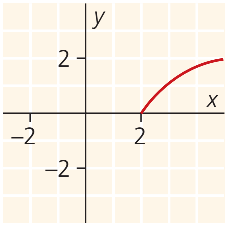

b.



c. 

d. 

e. 

f. 

**Communication** **2.** Explain what the derivative of a function represents.

**Communication** **3.** Illustrate two situations in which a function does not have a derivative at $x = 1$.

## Part B

**Knowledge/Understanding** **4.** For each function, find the value of the derivative $f'(a)$ for the given value of $a$.

a. $f(x) = x^2; a = 1$

b. $f(x) = x^2 + 3x + 1; a = 3$

c. $f(x) = \sqrt{x + 1}; a = 0$

5. Use the definition of the derivative to find $f'(x)$ for each function.

a. $f(x) = x^2 + 3x$ b. $f(x) = \frac{3}{x + 2}$ c. $f(x) = \sqrt{3x + 2}$ d. $f(x) = \frac{1}{x^2}$

6. In each case, find the derivative $\frac{dy}{dx}$.

a. $y = 6 - 7x$ b. $y = \frac{x + 1}{x - 1}$ c. $y = 3x^2$

7. Find the slope of the tangents to $y = 2x^2 - 4x$ when $x = 0$, $x = 1$, and $x = 2$. Sketch the graph, showing these tangents.

**Application** **8.** An object moves in a straight line with its position at time $t$ seconds given by $s(t) = -t^2 + 8t$ where $s$ is measured in metres. Find the velocity when $t = 0$, $t = 4$, and $t = 6$.

**Thinking/Inquiry/Problem Solving** **9.** Find an equation of the straight line that is tangent to the graph of $f(x) = \sqrt{x + 1}$ and parallel to $x - 6y + 4 = 0$.

10. For each function, use the definition of the derivative to determine $\frac{dy}{dx}$, where $a$, $b$, $c$, and $m$ are constants.

a. $y = c$  b. $y = x$  c. $y = mx + b$  d. $y = ax^2 + bx + c$

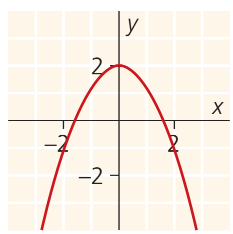
**Communication**

**11.** Does the function $f(x) = x^3$ ever have a negative slope? If so, where? Give reasons for your answer.

**Thinking/Inquiry/ Problem Solving**

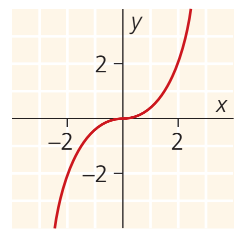
**12.** Match each function in graphs **a**, **b**, and **c** with its corresponding derivative, graphed in **d**, **e**, and **f.**

a.



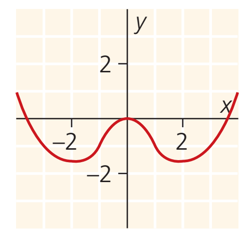

$y = f(x)$

b.



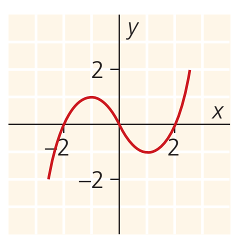

$y = f(x)$

c.



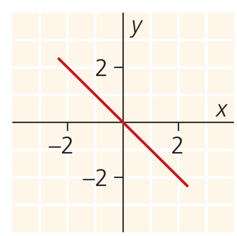

$y = f(x)$

d.



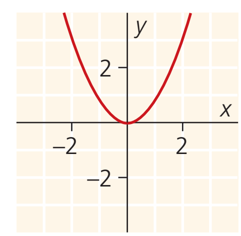

$y = f'(x)$

e.




$y = f'(x)$

f.




$y = f'(x)$

13. Find the slope of the tangent to the curve $\frac{1}{x} + \frac{1}{y} = 1$ at the point $(2, 2)$.

## Part C

14. For the function $f(x) = x|x|$, show that $f'(0)$ exists. What is the value?

**Application**

**15.** If $f(a) = 0$ and $f'(a) = 6$, find $\lim_{h \to 0} \frac{f(a+h)}{2h}$.

**Thinking/Inquiry/ Problem Solving**

**16.** Give an example of a function that is continuous on $-\infty < x < \infty$ but that is not differentiable at $x = 3$.

# Section 4.2 — The Derivatives of Polynomial Functions

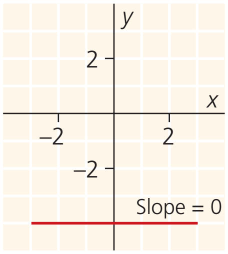
We have seen that derivatives of functions are of practical use because they represent instantaneous rates of change.

Computing derivatives from the definition, as we did in Section 4.1, is tedious and time-consuming. In this section, we will develop some rules that simplify the process of differentiation.

We will begin developing the rules of differentiation by looking at the constant function, $f(x) = k$. Since the graph of any constant function is a horizontal line with slope zero at each point, the derivative should be zero.

For example, if $f(x) = -4$ then $f'(3) = 0$.

Alternatively, we can write $\frac{d}{dx}(3) = 0$.




**The Constant Function Rule**

**If $f(x) = k$, where $k$ is a constant, then $f'(x) = 0$.**

**In Leibniz notation, $\frac{d}{dx}(k) = 0$.**

### Proof

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{k - k}{h} \qquad \text{(Since } f(x) = k \text{ and } f(x+h) = k \text{ for all } h\text{)} \\
&= \lim_{h\to 0} 0 \\
&= 0
\end{aligned}
$$

---

**EXAMPLE 1**

a. If $f(x) = 5, f'(x) = 0$.

b. If $y = \frac{\pi}{2}, \frac{dy}{dx} = 0$.

A **power function** is a function of the form $f(x) = x^n$, where $n$ is a real number. In the previous section, we observed that for $f(x) = x^2$, $f'(x) = 2x$; for $g(x) = \sqrt{x} = x^{\frac{1}{2}}$, $g'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$; and for $h(x) = \frac{1}{x^2} = x^{-2}$, $h'(x) = -2x^{-3}$. We also hypothesized that $\frac{d}{dx}(x^n) = nx^{n-1}$. In fact, this is true, and is called the **Power Rule**.

**The Power Rule**

**If $f(x) = x^n$, where $n$ is a real number, then $f'(x) = nx^{n-1}$.**

**In Leibniz notation, $\frac{d}{dx}(x^n) = nx^{n-1}$.**

---

**EXAMPLE 2**

a. If $f(x) = x^7$, then $f'(x) = 7x^6$.

b. If $g(x) = \frac{1}{x^3} = x^{-3}$, then $g'(x) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$.

c. If $s = t^{\frac{3}{2}}$, $\frac{ds}{dt} = \frac{3}{2}t^{\frac{1}{2}} = \frac{3}{2}\sqrt{t}$.

d. $\frac{d}{dx}(x) = 1x^{1-1} = x^0 = 1$

### Proof of the Power Rule

(Where $n$ is a positive integer.)
Using the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{where } f(x) = x^n$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-x)\left[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}\right]}{h}$$ (Factoring a difference of $n^{th}$ powers)

$$= \lim_{h \to 0} \left[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}\right]$$ (Reduce the fraction)

$$= (x)^{n-1} + x^{n-2}x + \ldots + (x)x^{n-2} + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + \ldots + x^{n-1} + x^{n-1}$$ (Since there are $n$ terms)

$$= nx^{n-1}.$$

The proof for any real number $n$ will be investigated in Chapter 8.

**The Constant Multiple Rule**

**If $f(x) = kg(x)$, where $k$ is a constant, then $f'(x) = kg'(x)$.**

**In Leibniz notation, $\frac{d}{dx}(ky) = k\frac{dy}{dx}$.**

### Proof of the Constant Multiple Rule

Let $f(x) = kg(x)$. By the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{kg(x+h) - kg(x)}{h} \\ &= \lim_{h \to 0} k\left[\frac{g(x+h) - g(x)}{h}\right] \\ &= k\lim_{h \to 0}\left[\frac{g(x+h) - g(x)}{h}\right] \qquad \text{(Property of Limits)} \\ &= kg'(x). \end{aligned}$$

---

**EXAMPLE 3** Differentiate the following:

a. $f(x) = 7x^3$

b. $y = 12x^{\frac{4}{3}}$

### Solution

a. $f(x) = 7x^3$

$$f'(x) = 7\frac{d}{dx}(x^3) = 7(3x^2) = 21x^2$$

b. $y = 12x^{\frac{4}{3}}$

$$\frac{dy}{dx} = 12\frac{d}{dx}\left(x^{\frac{4}{3}}\right) = 12\left(\frac{4}{3}x^{\frac{4}{3}-1}\right) = 16x^{\frac{1}{3}}$$

We conclude this section on Rules of Differentiation with the Sum and Difference Rules.

**The Sum Rule**

**If functions $p(x)$ and $q(x)$ are differentiable and $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.**

**In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$.**

### Proof of the Sum Rule

Let $f(x) = p(x) + q(x)$. By the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[p(x+h) + q(x+h)] - [p(x) + q(x)]}{} \\
&= \lim_{h\to 0} \left\{ \frac{[p(x+h) - p(x)]}{h} + \frac{[q(x+h) - q(x)]}{h} \right\} \\
&= \lim_{h\to 0} \left\{ \frac{[p(x+h) - p(x)]}{h} \right\} + \lim_{h\to 0} \left\{ \frac{[q(x+h) - q(x)]}{h} \right\} \qquad \text{(Sum Property of Limits)} \\
&= p'(x) + q'(x)
\end{aligned}$$

The proof of the Difference Rule is similar to that of the Sum Rule.

**The Difference Rule**
**If $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.**

---

**EXAMPLE 4** Differentiate the following functions:

a. $f(x) = 3x^2 - 5\sqrt{x}$

b. $y = (3x + 2)^2$

### Solution

We apply the Constant Multiple, Power, Sum, and Difference Rules.

a.
$$\begin{aligned}
f(x) &= 3x^2 - 5\sqrt{x} \\
f'(x) &= \frac{d}{dx}(3x^2) - \frac{d}{dx}\left(5x^{\frac{1}{2}}\right) \\
&= 3\frac{d}{dx}(x^2) - 5\frac{d}{dx}\left(x^{\frac{1}{2}}\right) \\
&= 3(2x) - 5\left(\frac{1}{2}x^{-\frac{1}{2}}\right) \\
&= 6x - \frac{5}{2}x^{-\frac{1}{2}}
\end{aligned}$$

b. We first expand $y = (3x + 2)^2$.

$$\begin{aligned}
y &= 9x^2 + 12x + 4 \\
\frac{dy}{dx} &= 18x + 12
\end{aligned}$$

**EXAMPLE 5** Find the equation of the tangent to the graph of $f(x) = -x^3 + 3x^2 - 2$ at $x = 1$.

technology

### Solution

The slope of the tangent to the graph of $f$ at any point is given by the derivative $f'(x)$.

For $\quad f(x) = -x^3 + 3x^2 - 2$

$\quad f'(x) = -3x^2 + 6x$

Now, $\quad f'(1) = -3(1)^2 + 6(1)$

$\quad = -3 + 6$

$\quad = 3.$

The slope of the tangent at $x = 1$ is 3.

The point of tangency is $(1, f(1)) = (1, 0)$.

The equation of the tangent is

$y - 0 = 3(x - 1)$ or $y = 3x - 3$.

### Solution Using the Graphing Calculator

Draw the graph using a graphing calculator and draw the tangent at $x = 1$. The point is $(1, 0)$. Input $y_1 = -x^3 + 3x^2 - 2$.

Select ZOOM 4.

Select 2nd PRGM (DRAW) for the DRAW program.

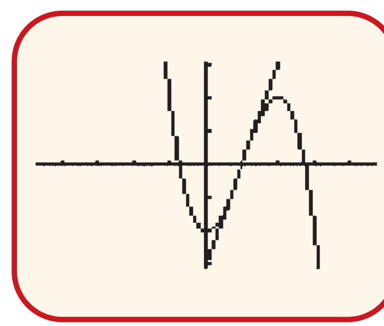
Select **5:Tangent(.**

Select VARS and then Y-VARS and **1:Function.**

Press ENTER for the $y_1$ function.

Complete the instructions so the window looks like Tangent $(y_1, 1)$.

Press ENTER to see the graph.



**EXAMPLE 6** Find the points on the graph in Example 5 where the tangents are horizontal.

### Solution

Horizontal lines have slope 0. We need to find the values of $x$ that satisfy $f'(x) = 0$.

$$-3x^2 + 6x = 0$$
$$-3x(x - 2) = 0$$
$$x = 0 \text{ or } x = 2$$

The graph of $f(x) = -x^3 + 3x^2 - 2$ has horizontal tangents at $(0, -2)$ and $(2, 2)$.

# Exercise 4.2

## Part A

Communication

**1.** What rules do you know for calculating derivatives? Give examples of each.

2. Find $f'(x)$ for each of the following:

a. $f(x) = 4x - 7$
b. $f(x) = 7$
c. $f(x) = 2x^2 + x - 5$
d. $f(x) = \sqrt{x}$
e. $f(x) = 4x^3 + 2$
f. $f(x) = x^3 - x^2$
g. $f(x) = -x^2 + 5x + 8$
h. $f(x) = \sqrt[3]{x}$
i. $f(x) = \frac{1}{4}x^4$
j. $f(x) = (3x)^2$
k. $f(x) = \left(\frac{x}{2}\right)^4$
l. $f(x) = x^{-3}$

Knowledge/Understanding

**3.** Differentiate each function. Use either the Leibniz notation or prime notation, depending on which is appropriate.

a. $y = x^2 - 3x + 1$
b. $f(x) = 2x^3 + 5x^2 - 4x - 3.75$
c. $v(t) = 6t^3 - 4t^5$
d. $s(t) = \frac{1}{t^2}, t > 0$
e. $f(x) = (x^3)^2$
f. $h(x) = (2x + 3)(x + 4)$
g. $s = t^2(t^2 - 2t)$
h. $g(x) = \frac{4}{x^{-5}}$
i. $y = \frac{1}{5}x^5 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 1$
j. $g(x) = 5(x^2)^4$
k. $s(t) = \frac{t^5 - 3t^2}{2t}, t > 0$
l. $g(x) = 7f(x) + 5$
m. $h(x) = -\frac{3}{x^7}$
n. $y = mx + b$

4. Apply the differentiation rules of this section to find the derivatives of the following:

a. $f(x) = 10x^{\frac{1}{5}}$
b. $y = 3x^{\frac{5}{3}}$
c. $y = 6x^{-\frac{3}{2}}$
d. $y = x^8 - x^{-8}$
e. $y = 3x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + x^{-\frac{1}{3}}$
f. $y = 4x^{-\frac{1}{2}} - \frac{6}{x}$
g. $y = \frac{6}{x^3} + \frac{2}{x^2} - 3$
h. $y = 9x^{-2} + 3\sqrt{x}$
i. $y = 20x^5 + 3\sqrt[3]{x} + 17$
j. $y = \sqrt{x} + 6\sqrt{x^3} + \sqrt{2}$
k. $y = x^{1.5} - 12x^{-0.25}$
l. $y = \frac{1 + \sqrt{x}}{x}$

## Part B

5. Let $s$ represent the position of a moving object at time $t$. Find the velocity $v = \frac{ds}{dt}$ at time $t$.

a. $s = -2t^2 + 7t$
b. $s = 18 + 5t - \frac{1}{3}t^3$
c. $s = (t - 3)^2$

6. Find $f'(a)$ for the given function $f(x)$ at the given value of $a$.

a. $f(x) = x^3 - \sqrt{x}$; $a = 4$

b. $f(x) = 7 - 6\sqrt{x} + 5x^{\frac{2}{3}}$; $a = 64$

Application

**7.** Find the slope of the tangent to each of the curves at the given point.

a. $y = 3x^4$ at $(1, 3)$

b. $y = \frac{1}{x^{-5}}$ at $(-1, -1)$

c. $y = \frac{2}{x}$ at $(-2, -1)$

d. $y = \sqrt{16x^3}$ at $(4, 32)$

8. Find the slope of the tangent to the graph of the function at the point whose $x$-coordinate is given.

a. $y = 2x^3 + 3x$; $x = 1$

b. $y = 2\sqrt{x} + 5$; $x = 4$

c. $y = \frac{16}{x^2}$; $x = -2$

d. $y = x^{-3}(x^{-1} + 1)$; $x = 1$

9. Write an equation of a tangent to each of the curves at the given point.

a. $y = 2x - \frac{1}{x}$ at $P(0.5, -1)$

b. $y = \frac{3}{x^2} - \frac{4}{x^3}$ at $P(-1, 7)$

c. $y = \sqrt{3x^3}$ at $P(3, 9)$

d. $y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right)$ at $P(1, 2)$

e. $y = \left(\sqrt{x} - 2\right)\left(3\sqrt{x} + 8\right)$ at $P(4, 0)$

f. $y = \frac{\sqrt{x} - 2}{\sqrt[3]{x}}$ at $P(1, -1)$

Communication

**10.** What is a normal to the graph of a function? Find the equation of the normal to the graph of the function in Question 9, part **b** at the given point.

Thinking/Inquiry/
Problem Solving

**11.** Find the values of $x$ so that the tangent to the function $y = \frac{3}{\sqrt[3]{x}}$ is parallel to the line $x + 16y + 3 = 0$.

Communication

**12.** Do the functions $y = \frac{1}{x}$ and $y = x^3$ ever have the same slope? If so, where?

13. Tangents are drawn to the parabola $y = x^2$ at $(2, 4)$ and $\left(-\frac{1}{8}, \frac{1}{64}\right)$. Prove that these lines are perpendicular. Illustrate with a sketch.

14. Find the point on the parabola $y = -x^2 + 3x + 4$ where the slope of the tangent is 5. Illustrate your answer with a sketch.

15. Find the coordinates of the points on the graph of $y = x^3 + 2$ at which the slope of the tangent is 12.

16. Show that there are two tangents to the curve $y = \frac{1}{5}x^5 - 10x$ that have a slope of 6.

**Application** **17.** Find the equations of the tangents to the curve $y = 2x^2 + 3$ that pass through the following:

a. point (2, 3)

b. point (2, −7)

18. Find the value of $a$, given that the line $ax - 4y + 21 = 0$ is tangent to the graph of $y = \frac{a}{x^2}$ at $x = -2$.

**Application** **19.** It can be shown that from a height of $h$ metres, a person can see a distance $d$ kilometres to the horizon, where $d = 3.53\sqrt{h}$.

a. When the elevator of the CN Tower passes the 200 m height, how far can the passengers in the elevator see across Lake Ontario?

b. Find the rate of change of this distance with respect to height when the height of the elevator is 200 m.

20. A subway train travels from one station to the next in 2 min. Its distance, in kilometres, from the first station after $t$ minutes is $s(t) = t^2 - \frac{1}{3}t^3$. At what times will the train have a velocity of 0.5 km/min?

21. A construction worker drops a bolt while working on a high-rise building 320 m above the ground. After $t$ seconds, the bolt has fallen a distance of $s$ metres, where $s(t) = 5t^2$, $0 \le t \le 8$. The function that gives the height of the bolt above ground at time $t$ is $R(t) = 320 - 5t^2$. Use this function to determine the velocity of the bolt at $t = 2$.

22. Tangents are drawn from the point (0, 3) to the parabola $y = -3x^2$. Find the coordinates of the points at which these tangents touch the curve. Illustrate your answer with a sketch.

23. The tangent to the cubic function defined by $y = x^3 - 6x^2 + 8x$ at point $A(3, -3)$ intersects the curve at another point, $B$. Find the coordinates of $B$. Illustrate with a sketch.

## Part C

**Thinking/Inquiry/ Problem Solving** **24.** Let $P(a, b)$ be a point on the curve $\sqrt{x} + \sqrt{y} = 1$. Show that the slope of the tangent at $P$ is $-\sqrt{\frac{b}{a}}$.

**Thinking/Inquiry/ Problem Solving** **25.** For the power function $f(x) = x^n$, find the $x$-intercept of the tangent to its graph at point (1, 1). What happens to the $x$-intercept as $n$ increases without bound ($n \to +\infty$)? Explain the result geometrically.

**Application** **26.** For each function, sketch the graph of $y = f(x)$ and find an expression for $f'(x)$. Indicate any points at which $f'(x)$ does not exist.

a. $f(x) = \begin{cases} x^2, x < 3 \\ x + 6, x \ge 3 \end{cases}$

b. $f(x) = |3x^2 - 6|$

c. $f(x) = ||x| - 1|$

# Section 4.3 — The Product Rule

In this section, we will develop a rule for differentiating the product of two functions, such as $f(x) = (3x^2 - 1)(x^3 + 8)$ and $g(x) = (x - 3)^3(x + 2)^2$, without first expanding the expressions.

You might suspect that the derivative of a product of two functions is simply the product of the separate derivatives. An example shows that this is not so.

---

**EXAMPLE 1**

Let $h(x) = f(x)g(x)$, where $f(x) = (x^2 + 2)$ and $g(x) = (x + 5)$.

Show that $h'(x) \neq f'(x)g'(x)$.

### Solution

The expression $h(x)$ can be simplified.

$$\begin{aligned} h(x) &= (x^2 + 2)(x + 5) \\ &= x^3 + 5x^2 + 2x + 10 \end{aligned}$$

The derivative of $h(x)$ is $h'(x) = 3x^2 + 10x + 2$.

The derivatives of the functions $f(x)$ and $g(x)$ are

$$f'(x) = 2x \text{ and } g'(x) = 1.$$

The product $f'(x)g'(x) = (2x)(1) = 2x$.

Since $2x$ is not the derivative of $h(x)$, we have shown that $h'(x) \neq f'(x)g'(x)$.

The correct method for differentiating a product of two functions uses the following rule.

**The Product Rule**

**If $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$.**

**If $u$ and $v$ are functions of $x$, $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$.**

In words, the Product Rule says, "the derivative of the product of two functions is equal to the derivative of the first function times the second function plus the first function times the derivative of the second function."

### Proof of the Product Rule

$h(x) = f(x)g(x)$; then using the definition of the derivative,

$$h'(x) = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}.$$

In Section 4.1, we saw that $f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$,

and $g'(x) = \lim_{h \to 0}\left[\frac{g(x+h) - g(x)}{h}\right]$. To evaluate $h'(x)$ we subtract and add terms in the numerator.

$$\begin{aligned}
\text{Now } h'(x) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0}\left\{\left[\frac{f(x+h) - f(x)}{h}\right]g(x+h) + f(x)\left[\frac{g(x+h) - g(x)}{h}\right]\right\} \\
&= \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]\lim_{h \to 0} g(x+h) + \lim_{h \to 0} f(x)\lim_{h \to 0}\left[\frac{g(x+h) - g(x)}{h}\right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

---

**EXAMPLE 2** Differentiate $h(x) = (x^2 - 3x)(x^5 + 2)$, using the Product Rule.

### Solution

$$h(x) = (x^2 - 3x)(x^5 + 2)$$

Using the Product Rule, we get

$$\begin{aligned}
h'(x) &= (2x - 3)(x^5 + 2) + (x^2 - 3x)(5x^4) \\
&= 2x^6 - 3x^5 + 4x - 6 + 5x^6 - 15x^5 \\
&= 7x^6 - 18x^5 + 4x - 6.
\end{aligned}$$

We can, of course, differentiate the function after we first expand. The Product Rule will be essential, however, when we work with products of polynomials such as $f(x) = (x^2 + 9)(x^3 + 5)^4$ or non-polynomial functions such as $f(x) = (x^2 + 9)\sqrt{x^3 + 5}$.

It is not necessary to simplify the expression when you are asked to calculate the derivative at a particular value of $x$. Because many expressions derived using differentiation rules are cumbersome, it is easier to substitute, as in the preceding example.

The next example can be determined by multiplying out the two polynomials and then calculating the derivative of the resulting polynomial at $x = -1$. Instead we shall apply the Product Rule.

**EXAMPLE 3** Find the value $h'(-1)$ for the function $h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$.

**Solution**

$h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$

Using the Product Rule, we get

$$\begin{aligned} h'(x) &= (15x^2 + 14x)(2x^2 + x + 6) + (5x^3 + 7x^2 + 3)(4x + 1) \\ h'(-1) &= [15(-1)^2 + 14(-1)][2(-1)^2 + (-1) + 6] + [5(-1)^3 \\ &\quad + 7(-1)^2 + 3][4(-1) + 1] \\ &= (1)(7) + (5)(-3) \\ &= -8. \end{aligned}$$

The following example illustrates the extension of the Product Rule to more than two functions.

**EXAMPLE 4** Find an expression for $p'(x)$ if $p(x) = f(x)g(x)h(x)$.

**Solution**

We temporarily regard $f(x)g(x)$ as a single function.

$$p(x) = [f(x)g(x)]h(x)$$

By the Product Rule,

$$p'(x) = [f(x)g(x)]'h(x) + [f(x)g(x)]h'(x).$$

A second application of the Product Rule yields

$$\begin{aligned} p'(x) &= [f'(x)g(x) + f(x)g'(x)]h(x) + f(x)g(x)h'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x). \end{aligned}$$

This expression gives us the **Extended Product Rule** for the derivative of the product of three functions. Its symmetric form makes it easy to extend the process to the product of four or more functions.

## The Power of a Function Rule for Positive Integers

Suppose that we now wish to differentiate functions such as

$y = (x^2 - 3)^4$ or $y = (x^2 + 3x + 5)^6$.

These functions are of the form $y = u^n$, where $n$ is a positive integer and $u = g(x)$ is a function whose derivative we can find. Using the Product Rule, we can develop an efficient method for differentiating such functions.

For the case $n = 2$, $h(x) = [g(x)]^2$

$$h(x) = g(x)g(x)$$

and using the Product Rule,

$$h'(x) = g'(x)g(x) + g(x)g'(x)$$
$$= 2g'(x)g(x).$$

Similarly, for $n = 3$, we can use the Extended Product Rule.

Thus $h(x) = [g(x)]^3$

$$= g(x)g(x)g(x)$$
$$h'(x) = g'(x)g(x)g(x) + g(x)g'(x)g(x) + g(x)g(x)g'(x)$$
$$= 3[g(x)]^2g'(x).$$

These results suggest a generalization of the Power Rule.

**The Power of a Function Rule for Positive Integers**

**If $u$ is a function of $x$, and $n$ is a positive integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$.**

**In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1}g'(x)$.**

The Power of a Function Rule is a *special case* of the Chain Rule, which we will discuss later in this chapter. We are now able to differentiate any polynomial, such as $h(x) = (x^2 + 3x + 5)^6$ or $h(x) = (1 - x^2)^4(2x + 6)^3$, without multiplying out the brackets.

---

**EXAMPLE 5**

For $h(x) = (x^2 + 3x + 5)^6$, find $h'(x)$.

**Solution**

Here $h(x)$ has the form $h(x) = [g(x)]^6$, where the "inside" function is $g(x) = x^2 + 3x + 5$.

By the Power of a Function Rule, we get $h'(x) = 6(x^2 + 3x + 5)^5(2x + 3)$.

---

**EXAMPLE 6**

The position $s$, in centimetres, of an object moving in a straight line is given by $s = t(6 - 3t)^4$, $t \geq 0$, where the time $t$ is in seconds. Determine its velocity at time $t = 2$.

**Solution**

The velocity of the object at any time $t \geq 0$ is $v = \frac{ds}{dt}$.

$$
\begin{aligned}
v &= \frac{d}{dt}\Big[t(6-3t^4)\Big] \\
&= (1)(6-3t)^4 + (t)\frac{d}{dt}\Big[(6-3t)^4\Big] && \text{(Product Rule)} \\
&= (6-3t)^4 + (t)\Big[4(6-3t)^3(-3)\Big] && \text{(Power of a Function Rule)}
\end{aligned}
$$

At $t = 2$,

$$
\begin{aligned}
v &= 0 + (2)[4(0)(-3)] \\
&= 0.
\end{aligned}
$$

We conclude that the object is at rest at time $t = 2$.

Notice that if the derivative is required at a particular value of the independent variable, it is not necessary to simplify before substituting.

# Exercise 4.3

## Part A

1. Use the Product Rule to differentiate each function. Simplify your answers.
   a. $h(x) = x(x-4)$
   b. $h(x) = x^2(2x-1)$
   c. $h(x) = (3x+2)(2x-7)$
   d. $h(x) = (8-x)(4x+6)$
   e. $h(x) = (5x^7+1)(x^2-2x)$
   f. $s(t) = (t^2+1)(3-2t^2)$

Knowledge/Understanding

2. Use the Product Rule and the Power of a Function Rule to differentiate the following. Do not simplify.
   a. $y = (5x+1)^3(x-4)$
   b. $y = (3x^2+4)(3+x^3)^5$
   c. $y = (1-x^2)^4(2x+6)^3$

Communication

3. When is it not appropriate to use the Product Rule? Give examples.

## Part B

4. Find the value of $\frac{dy}{dx}$ for the given value of $x$.
   a. $y = (2+7x)(x-3)$, $x = 2$
   b. $y = (1-2x)(1+2x)$, $x = \frac{1}{2}$
   c. $y = (3-2x-x^2)(x^2+x-2)$, $x = -2$
   d. $y = (4x^2+2x)(3-2x-5x^2)$, $x = 0$
   e. $y = x^3(3x+7)^2$, $x = -2$

f. $y = (2x + 1)^5(3x + 2)^4$, $x = -1$

g. $y = x(5x - 2)(5x + 2)$, $x = 3$

h. $y = 3x(x - 4)(x + 3)$, $x = 2$

5. Find the equation of the tangent to the curve $y = (x^3 - 5x + 2)(3x^2 - 2x)$ at the point $(1, -2)$.

6. Find the point(s) where the tangent to the curve is horizontal.

   a. $y = 2(x - 29)(x + 1)$
   
   b. $y = (x^2 + 2x + 1)(x^2 + 2x + 1)$

7. Use the Extended Product Rule to differentiate the following. Do not simplify.

   a. $y = (x + 1)^3(x + 4)(x - 3)^2$
   
   b. $y = x^2(3x^2 + 4)^2(3 - x^3)^4$

Communication

8. Find the slope of the tangent to $h(x) = 2x(x + 1)^3(x^2 + 2x + 1)^2$ at $x = -2$. Explain how to find the equation of the normal at $x = -2$.

## Part C

Thinking/Inquiry/Problem Solving

9. a. Find an expression for $f'(x)$ if $f(x) = g_1(x)g_2(x)g_3(x) \ldots g_{n-1}(x)g_n(x)$.

   b. If $f(x) = (1 + x)(1 + 2x)(1 + 3x) \ldots (1 + nx)$, find $f'(0)$.

10. Determine a quadratic function $f(x) = ax^2 + bx + c$ whose graph passes through the point $(2, 19)$ and that has a horizontal tangent at $(-1, -8)$.

11. Sketch the graph of $f(x) = |x^2 - 1|$.

    a. For what values of $x$ is $f$ not differentiable?
    
    b. Find a formula for $f'$ and sketch the graph of $f'$.
    
    c. Find $f'$ at $x = -2$, 0, and 3.

12. Show that the line $4x - y + 11 = 0$ is tangent to the curve $y = \frac{16}{x^2} - 1$.

# Section 4.4 — The Quotient Rule

In the previous section, we found that the derivative of the product of two functions is not the product of their derivatives. The Quotient Rule gives the derivatives of a function that is the quotient of two functions. It is derived from the Product Rule.

**The Quotient Rule**

**If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.**

**In Leibniz notation, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$.**

## Memory Aid for the Product and Quotient Rules

It is worth noting that the Quotient Rule is similar to the Product Rule in that both have $f'(x)g(x)$ and $f(x)g'(x)$. For the Product Rule, we put an addition sign $(+)$ between both terms. For the Quotient Rule, we put a subtraction sign $(-)$ between the terms and then divide the result by the square of the original denominator.

### Proof of the Quotient Rule

We wish to find $h'(x)$, given that

$$h(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0.$$

We rewrite this as a product:

$$h(x)g(x) = f(x).$$

Using the Product Rule,

$$h'(x)g(x) + h(x)g'(x) = f'(x).$$

Solving for $h'(x)$, we get

$$\begin{aligned} h'(x) &= \frac{f'(x) - h(x)g'(x)}{g(x)} \\ &= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \end{aligned}$$

The Quotient Rule enables us to differentiate rational functions.

**EXAMPLE 1** Find the derivative of $h(x) = \dfrac{3x - 4}{x^2 + 5}$.

### Solution

Since $h(x) = \dfrac{f(x)}{g(x)}$, where $f(x) = 3x - 4$ and $g(x) = x^2 + 5$, we use the Quotient Rule to find $h'(x)$.

Using the Quotient Rule, we get
$$\begin{aligned} h'(x) &= \frac{(3)(x^2 + 5) - (3x - 4)(2x)}{(x^2 + 5)^2} \\ &= \frac{3x^2 + 15 - 6x^2 + 8x}{(x^2 + 5)^2} \\ &= \frac{-3x^2 + 8x + 15}{(x^2 + 5)^2}. \end{aligned}$$

**EXAMPLE 2** Using a graphing calculator, graph $y = \dfrac{2x}{x^2 + 1}$ and the tangent to it at $x = 0$. Find the equation of the tangent.

technology

### Solution

The slope of the tangent to the graph of $y$ at any point is given by the derivative $\dfrac{dy}{dx}$.

By the Quotient Rule,

$$\frac{dy}{dx} = \frac{(2)(x^2 + 1) - (2x)(2x)}{(x^2 + 1)^2}.$$

At $x = 0$,

$$\frac{dy}{dx} = \frac{(2)(0 + 1) - (0)(0)}{(0 + 1)^2} = 2.$$

Since the slope of the tangent at $x = 0$ is 2 and the point of tangency is (0, 0), the equation of the tangent is $y = 2x$.

### Solution Using the Calculator

Use the DRAW function to draw a tangent at the point (0, 0).

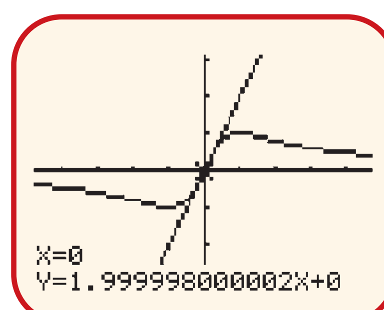
In this case, input $y_1 = \dfrac{2x}{x^2 + 1}$.

Select ZOOM **4:ZDecimal** for your domain and range.

Select GRAPH.

Select 2nd PRGM (DRAW) and choose **5:Tangent(** to obtain the graph window.

Select $x = 0$ and then press ENTER to graph the tangent.



**EXAMPLE 3** Find the coordinates of each point on the graph of $f(x) = \frac{2x + 8}{\sqrt{x}}$ where the tangent is horizontal.

**Solution**

The slope of the tangent at any point on the graph is given by $f'(x)$.
Using the Quotient Rule,

$$\begin{aligned} f'(x) &= \frac{(2)(\sqrt{x}) - (2x + 8)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{(\sqrt{x})^2} \\ &= \frac{2\sqrt{x} - \frac{2x + 8}{2\sqrt{x}}}{x} \\ &= \frac{\frac{2x}{\sqrt{x}} - \frac{x + 4}{\sqrt{x}}}{x} \\ &= \frac{x - 4}{\phantom{x}}. \end{aligned}$$

The tangent will be horizontal when $f'(x) = 0$, that is, when $x = 4$. The point on the graph where the tangent is horizontal is $(4, 8)$.

# Exercise 4.4

## Part A

Communication **1.** What are the Exponential Rules? Give examples of each rule.

2. Copy and complete the table *without* using the Quotient Rule.

| Function | Rewrite | Differentiate and simplify, if necessary |
|---|---|---|
| $f(x) = \frac{x^2 + 3x}{x}, x \neq 0$ | | |
| $g(x) = \frac{3x^{\frac{5}{3}}}{x}, x \neq 0$ | | |
| $h(x) = \frac{1}{10x^5}, x \neq 0$ | | |
| $y = \frac{8x^3 + 6x}{2x}, x \neq 0$ | | |
| $s = \frac{t^2 - 9}{t - 3}, t \neq 3$ | | |

Communication **3.** What rule do we use to find the derivative of a rational function?

## Part B

**Knowledge/Understanding**

**4.** Use the Quotient Rule to differentiate each function. Simplify your answers.

a. $h(x) = \dfrac{x}{x+1}$ b. $h(x) = \dfrac{x^2}{x+1}$ c. $h(t) = \dfrac{2t-3}{t+5}$

d. $h(x) = \dfrac{2x-1}{x+3}$ e. $h(x) = \dfrac{x^3}{2x^2-1}$ f. $h(x) = \dfrac{1}{x^2+3}$

g. $y = \dfrac{x(3x+5)}{1-x^2}$ h. $y = \dfrac{x^2-x+1}{x^2+3}$ i. $y = \dfrac{x^2-1}{x(3x+1)}$

5. Find $\dfrac{dy}{dx}$ at the given value of $x$.

a. $y = \dfrac{3x+2}{x+5}$, $x = -3$ b. $y = \dfrac{x^3}{x^2+9}$, $x = 1$

c. $y = \dfrac{x^2-25}{x^2+25}$, $x = 2$ d. $y = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$, $x = 4$

6. Find the slope of the tangent to the curve $y = \dfrac{x^3}{x^2-6}$ at point $(3, 9)$.

7. Find the points on the graph of $y = \dfrac{3x}{x-4}$, where the slope of the tangent is $-\dfrac{12}{25}$.

**Thinking/Inquiry/Problem Solving**

**8.** Show that there are no tangents to the graph of $f(x) = \dfrac{5x+2}{x+2}$ that have a negative slope.

9. Find the point(s) at which the tangent to the curve is horizontal.

a. $y = \dfrac{2x^2}{x-4}$ b. $y = \dfrac{x^2-1}{x^2+x-2}$

**Application**

**10.** An initial population, $p$, of 1000 bacteria grows in number according to the equation $p(t) = 1000\left(1 + \dfrac{4t}{t^2+50}\right)$, where $t$ is in hours. Find the rate at which the population is growing after 1 h and after 2 h.

**Application**

**11.** Determine the equation of the tangent to the curve $y = \dfrac{x^2-1}{3x}$ at $x = 2$.

12. A motorboat coasts toward a dock with its engine off. Its distance $s$, in metres, from the dock $t$ seconds after the engine is turned off is

$s(t) = \dfrac{10(6-t)}{t+3}$ for $0 \leq t \leq 6$.

a. How far is the boat from the dock initially?

b. Find the boat's velocity when it bumps into the dock.

## Part C

**Thinking/Inquiry/Problem Solving**

**13.** Consider the function $f(x) = \dfrac{ax+b}{cx+d}$, $x \neq -\dfrac{d}{c}$, where $a$, $b$, $c$, and $d$ are non-zero constants. What condition on $a$, $b$, $c$, and $d$ ensures that each tangent to the graph of $f$ has positive slope?

# Section 4.5 — Composite Functions

Many functions can be thought of as the combination of simpler functions. If $f(x) = x^3$ and $g(x) = 2x$, then $h(x) = x^3 + 2x$ can be expressed as $h(x) = f(x) + g(x)$. It is often helpful to regard complicated functions as being built from simpler ones.

For example, if

$y = h(x) = (2x - 1)^5$,

we can consider this as the fifth power of $2x - 1$. That is,

$y = u^5$, where $u = 2x - 1$.

If we let $f(u) = u^5$ and $g(x) = 2x - 1$,

then $\quad y = h(x) = f(g(x))$

$\qquad\qquad = (2x - 1)$.

As another example, consider

$$h(x) = \sqrt{25 - x^2},$$

which can be considered to be made up of

$$g(x) = 25 - x^2 \text{ and } f(x) = \sqrt{x},$$

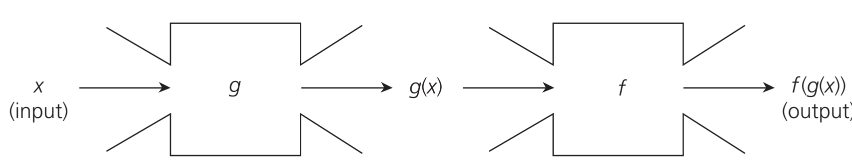
and once again,

$$h(x) = f(g(x)) = \sqrt{25 - x^2}.$$

This process of combining functions is called **composition**. We start with a number $x$ in the domain of $g$, find its image $g(x)$, and then take the value of $f$ at $g(x)$, providing $g(x)$ is in the domain of $f$. The result is the new function $h(x) = f(g(x))$ which is called the **composite function** of $f$ and $g$ and is denoted $f \circ g$.

**Given two functions $f$ and $g$, the composite function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.**

The domain of $f \circ g$ is the set of all $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$. We say that $g$ is the **inner** function and that $f$ is the **outer** function. A good way to picture the composition $f \circ g$ is by imagining a machine.



**EXAMPLE 1** If $f(x) = \sqrt{x}$ and $g(x) = x + 5$, find each of the following:

a. $f(g(4))$ b. $g(f(4))$ c. $f(g(x))$ d. $g(f(x))$

**Solution**

a. Since $g(4) = 9$, we have $f(g(4)) = f(9) = 3$.

b. Since $f(4) = 2$, we have $g(f(4)) = g(2) = 7$. *Note:* $f(g(4)) \neq g(f(4))$.

c. $f(g(x)) = f(x + 5) = \sqrt{x + 5}$

d. $g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{x} + 5$ *Note:* $f(g(x)) \neq g(f(x))$.

**EXAMPLE 2** We are given that $h(x) = f(g(x)) = \frac{1}{1 + x^2}$.

a. If $f(x) = \frac{1}{x}$, what is $g(x)$?

b. If $g(x) = x^2$, what is $f(x)$?

**Solution**

a. Since $f(x) = \frac{1}{x}$, then $f(g(x)) = \frac{1}{g(x)} = \frac{1}{1 + x^2}$. We conclude that $g(x) = 1 + x^2$.

b. Since $g(x) = x^2$, then $f(g(x)) = f(x^2) = \frac{1}{1 + x^2}$. Since $x^2$ has been substituted for $x$ in $f$, we conclude that $f(x) = \frac{1}{1 + x}$.

# Exercise 4.5

## Part A

1. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, find the following:

   a. $f(g(1))$ b. $g(f(1))$ c. $g(f(0))$

   d. $f(g(-4))$ e. $f(g(x))$ f. $g(f(x))$

2. For each of the following pairs of functions, find the composite functions $f \circ g$ and $g \circ f$. What is the domain of each composite function? Are the composite functions equal?

   a. $f(x) = x^2$, $g(x) = \sqrt{x}$ b. $f(x) = \frac{1}{x}$, $g(x) = x^2 + 1$ c. $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x + 2}$

## Part B

3. Use the functions $f(x) = 3x + 1$, $g(x) = x^3$, $h(x) = \frac{1}{x + 1}$, and $u(x) = \sqrt{x}$ to find expressions for the indicated composite function.

a. $f \circ u$ b. $u \circ h$ c. $g \circ f$

d. $u \circ g$ e. $h \circ u$ f. $f \circ g$

g. $h \circ (f \circ u)$ h. $(f \circ g) \circ u$ i. $g \circ (h \circ u)$

4. Express $h$ as the composition of two functions $f$ and $g$, such that $h(x) = f(g(x))$.

   a. $h(x) = (2x^2 - 1)^4$ b. $h(x) = \sqrt{5x - 1}$

   c. $h(x) = \dfrac{1}{x - 4}$ d. $h(x) = (2 - 3x)^{\frac{5}{2}}$

   e. $h(x) = x^4 + 5x^2 + 6$ f. $h(x) = (x + 1)^2 - 9(x + 1)$

5. If $f(x) = \sqrt{2 - x}$ and $f(g(x)) = \sqrt{2 - x^3}$, then what is $g(x)$?

6. If $g(x) = \sqrt{x}$ and $f(g(x)) = (\sqrt{x} + 7)^2$, then what is $f(x)$?

7. Let $g(x) = x - 3$. Find a function $f$ so that $f(g(x)) = x^2$.

8. Let $f(x) = x^2$. Find a function $g$ so that $f(g(x)) = x^2 + 8x + 16$.

9. Let $f(x) = x + 4$ and $g(x) = (x - 2)^2$. Find a function $u$ so that $f(g(u(x))) = 4x^2 - 8x + 8$.

10. If $f(x) = \dfrac{1}{1 - x}$ and $g(x) = 1 - x$, determine

    a. $g(f(x))$ b. $f(g(x))$

11. If $f(x) = 3x + 5$ and $g(x) = x^2 + 2x - 3$, determine $x$ such that $f(g(x)) = g(f(x))$.

12. If $f(x) = 2x - 7$ and $g(x) = 5 - 2x$,

    a. determine $f \circ f^{-1}$ and $f^{-1} \circ f$.

    b. show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

# Section 4.6 — The Derivative of a Composite Function

The Chain Rule tells us how to compute the derivative of the composite function $h(x) = f(g(x))$ in terms of the derivatives $f$ and $g$.

**The Chain Rule**

**If $f$ and $g$ are functions having derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by**

**$h'(x) = f'(g(x))g'(x)$.**

In words, the Chain Rule says, "the derivative of a composite function is the product of the derivative of the outer function evaluated at the inner function and the derivative of the inner function."

---

**EXAMPLE 1** Differentiate $h(x) = \left(x^2 + x\right)^{\frac{3}{2}}$.

### Solution

The inner function is $g(x) = x^2 + x$ and the outer function is $f(x) = x^{\frac{3}{2}}$.
The derivative of the inner function is $g'(x) = 2x + 1$.

The derivative of the outer function is $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$.

The derivative of the outer function evaluated at the inner function $g(x)$ is
$f'(x^2 + x) = \frac{3}{2}\left(x^2 + x\right)^{\frac{1}{2}}$.

By the Chain Rule,
$h'(x) = \frac{3}{2}\left(x^2 + x\right)^{\frac{1}{2}}(2x + 1)$.

### Proof of the Chain Rule

By the definition of the derivative, $[f(g(x))]' = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$.

Assuming $g(x + h) - g(x) \neq 0$, then we can write

$$[f(g(x))]' = \lim_{h \to 0}\left[\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right)\left(\frac{g(x+h) - g(x)}{h}\right)\right]$$

$$= \lim_{h \to 0}\left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right]\left[\lim_{h \to 0}\frac{g(x+h) - g(x)}{h}\right].$$

(Property of Limits)

Since $\lim_{h \to 0}[g(x+h) - g(x)] = 0$, let $g(x+h) - g(x) = k$ and $k \to 0$ as $h \to 0$, we obtain

$$[f(g(x))]' = \lim_{h\to 0}\left[\frac{f(g(x)+k)) - f(g(x))}{}\right]\left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{}\right].$$

Therefore, $\quad [f(g(x))]' = f'(g(x))g'(x).$

This proof is not valid for all circumstances, since dividing by $g(x+h) - g(x)$, we assume $g(x+h) - g(x) \neq 0$. A more advanced proof can be found in advanced calculus texts.

**The Chain Rule in Leibniz Notation**

**If $y = f(u)$, where $u = g(x)$, then $y$ is a composite function and**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

If we interpret derivatives as rates of change, the Chain Rule states that if $y$ is a function of $x$ through the intermediate variable $u$, then the rate of change of $y$ with respect to $x$ is equal to the product of the rate of change of $y$ with respect to $u$ and the rate of change of $u$ with respect to $x$.

---

**EXAMPLE 2** If $y = u^3 - 2u + 1$, where $u = 2\sqrt{x}$, find $\frac{dy}{dx}$ at $x = 4$.

**Solution**

Using the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = \left(3u^2 - 2\right)\left[2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\right]\\ &= \left(3u^2 - 2\right)\left(\frac{1}{\sqrt{x}}\right).\end{aligned}$$

It is not necessary to write the derivative entirely in terms of $x$.

When $x = 4$, $u = 2\sqrt{4} = 4$ and

$$\frac{dy}{dx} = \left[3(4^2) - 2\right]\left(\frac{1}{\sqrt{4}}\right) = (46)\left(\frac{1}{2}\right) = 23.$$

---

**EXAMPLE 3** An environmental study of a certain suburban community suggests that the average daily level of carbon monoxide in the air may be modelled by the formula

$C(p) = \sqrt{0.5p^2 + 17}$ parts per million when the population is $p$ thousand. It is estimated that $t$ years from now, the population of the community will be $p(t) = 3.1 + 0.1t^2$ thousand. At what rate will the carbon monoxide level be changing with respect to time three years from now?

### Solution

We are asked to find the value of $\frac{dC}{dt}$, when $t = 3$.

We can find the value of change by using the Chain Rule.

Therefore,
$$\frac{dC}{dt} = \frac{dC}{dP}\frac{dP}{dt}$$
$$= \left[\frac{1}{2}\left(0.5p^2 + 17\right)^{-\frac{1}{2}}(0.5)(2p)\right](0.2t).$$

When $t = 3$,
$$p(3) = 3.1 + 0.1(3)^2 = 4,$$

so
$$\frac{dC}{dt} = \left[\frac{1}{2}\left(0.5(4)^2 + 17\right)^{-\frac{1}{2}}(0.5)(2(4))\right](0.2(3))$$
$$= 0.24.$$

The carbon monoxide level will be changing at the rate of 0.24 parts per million. This level will be increasing because the sign of $\frac{dC}{dt}$ is positive.

---

**EXAMPLE 4** If $y = (x^2 - 5)^7$, find $\frac{dy}{dx}$.

### Solution

The inner function is $g(x) = x^2 - 5$ and the outer function is $f(x) = x^7$.
By the Chain Rule,
$$\frac{dy}{dx} = 7(x^2 - 5)^6(2x)$$
$$= 14x(x^2 - 5)^6.$$

Example 4 is a special case of the Chain Rule in which the outer function is a power function of the form $y = [g(x)]^n$. This leads to a generalization of the Power Rule.

**Power of a Function Rule**

**If $n$ is a real number and $u = g(x)$,**

**then** $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

**or** $$\frac{d}{dx}\left[g(x)\right]^n = n\left[g(x)\right]^{n-1}g'(x).$$

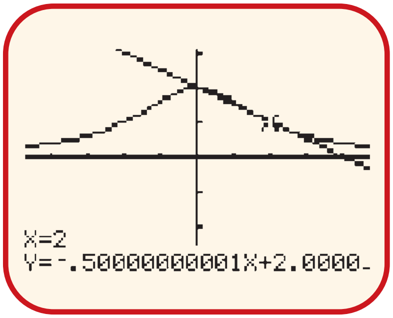
---

**EXAMPLE 5** technology

Using graphing technology, sketch the graph of the function $f(x) = \frac{8}{x^2 + 4}$.
Find the equation of the tangent at the point (2, 1) on the graph.

### Solution

From graphing technology, the graph is




The slope of the tangent at point (2, 1) is given by $f'(2)$.

We first write the function as $f(x) = 8(x^2 + 4)^{-1}$.

By the Power of a Function Rule, $\quad f'(x) = -8(x^2 + 4)^{-2}(2x)$.

The slope at (2, 1) is

$$\begin{aligned} f'(2) &= -8(4 + 4)^{-2}(4) \\ &= -\frac{32}{(8)^2} \\ &= -0.5. \end{aligned}$$

The equation of the tangent is $y - 1 = -\frac{1}{2}(x - 2)$ or $x + 2y - 4 = 0$.

---

**EXAMPLE 6** If $y = f(3x^4)$, and $f'(3) = -4$, find $\frac{dy}{dx}$ at $x = 1$.

### Solution

Here the inner function is $g(x) = 3x^4$.

By the Chain Rule, $\quad \frac{dy}{dx} = f'(3x^4)\left[12x^3\right]$.

When $x = 1$,

$$\begin{aligned} \frac{dy}{dx} &= f'(3)\left[12(1)^3\right] \\ &= (-4)(12) \\ &= -48. \end{aligned}$$

---

**EXAMPLE 7** Differentiate $h(x) = \sqrt[3]{\frac{x}{1-2x}}$. Express your answer in a simplified factored form.

### Solution

We write $h(x) = \left(\frac{x}{1-2x}\right)^{\frac{1}{3}}$, where $u = \frac{x}{1-2x}$ is the inner function and $u^{\frac{1}{3}}$ is the outer function.

Then

$$\begin{aligned} h'(x) &= \frac{1}{3}\left(\frac{x}{1-2x}\right)^{\frac{1}{3}-1}\frac{d}{dx}\left(\frac{x}{1-2x}\right) && \text{(Chain Rule)} \\ &= \frac{1}{3}\left(\frac{x}{1-2x}\right)^{-\frac{2}{3}}\left[\frac{(1)(1-2x)-(x)(-2)}{(1-2x)^2}\right] && \text{(Quotient Rule)} \\ &= \frac{1}{3}\left(\frac{x}{1-2x}\right)^{-\frac{2}{3}}\left[\frac{1}{(1-2x)^2}\right] \\ &= \frac{1}{3x^{\frac{2}{3}}(1-2x)^{\frac{4}{3}}}. \end{aligned}$$

# Exercise 4.6

## Part A

**Communication**

**1.** What is the rule for calculating the derivative of the composition of two differentiable functions? Give examples and show how the derivative is determined.

2. Differentiate each function. Do not expand any expression before differentiating.

a. $f(x) = (2x + 3)^4$
b. $y = (5 - x)^6$
c. $g(x) = (x^2 - 4)^3$
d. $y = (7 - x^3)^5$
e. $h(x) = (2x^2 + 3x - 5)^4$
f. $y = (5x - x^2)^5$
g. $f(x) = (\pi^2 - x^2)^3$
h. $y = (1 - x + x^2 - x^3)^4$
i. $f(x) = [(2 - x)^4 + 16]^3$
j. $g(x) = (4x + 1)^{\frac{1}{2}}$
k. $h(x) = \sqrt{5x + 7}$
l. $y = \sqrt{x^2 - 3}$
m. $f(x) = \dfrac{1}{(x^2 - 16)^5}$
n. $y = \dfrac{1}{\quad}$
o. $h(x) = \dfrac{1}{\quad}$
p. $f(u) = \left(1 + u^{\frac{1}{3}}\right)^6$
q. $y = \sqrt{(2x - 5)^3}$
r. $y = \left(\dfrac{x + 2}{\sqrt[3]{x}}\right)^3$

## Part B

**Knowledge/Understanding**

**3.** Rewrite each of the following in the form $y = u^n$ or $y = ku^n$, and then differentiate.

a. $y = \dfrac{3}{x^2}$
b. $y = -\dfrac{2}{x^3}$
c. $y = \dfrac{1}{x + 1}$
d. $y = \dfrac{1}{x^2 - 4}$
e. $y = \left(\dfrac{2}{x}\right)^2$
f. $y = \dfrac{3}{9 - x^2}$
g. $y = \dfrac{1}{5x^2 + x}$
h. $y = \dfrac{1}{(x^2 + x + 1)^4}$
i. $y = \left(\dfrac{1 + \sqrt{x}}{\sqrt[3]{x^2}}\right)^3$

4. Differentiate each function. Express your answer in a simplified factored form.

a. $f(x) = (x + 4)^3(x - 3)^6$
b. $y = \dfrac{3x + 5}{1 - x^2}$
c. $g(x) = (2x - 1)^4(2 - 3x)^4$
d. $y = \dfrac{3x^2 + 2x}{x^2 + 1}$
e. $h(x) = x^3(3x - 5)^2$
f. $y = \dfrac{(2x - 1)^2}{(x - 2)^3}$
g. $y = x^4(1 - 4x^2)^3$
h. $y = \left(\dfrac{x^2 - 3}{x^2 + 3}\right)^4$

i. $y = (x^2 + 3)^3(x^3 + 3)^2$

j. $h(x) = \dfrac{x}{\quad}$

k. $s = (4 - 3t^3)^4(1 - 2t)^6$

l. $h(x) = \dfrac{\sqrt{1 - x^2}}{\quad}$

5. Find the rate of change of each function at the given value of $t$. Leave your answers as rational numbers, or in terms of roots and the number $\pi$.

   a. $s(t) = t^{\frac{1}{3}}(4t - 5)^{\frac{2}{3}}, \ t = 8$

   b. $s(t) = \left(\dfrac{t - \pi}{t + 6\pi}\right)^{\frac{1}{3}}, \ t = 2\pi$

6. For what values of $x$ do the curves $y = (1 + x^3)^2$ and $y = 2x^6$ have the same slope?

7. Find the slope of the tangent to the curve $y = (3x + x^2)^{-2}$ at $\left(-2, \dfrac{1}{4}\right)$.

8. Find the equation of the tangent to the curve $y = (x^3 - 7)^5$ at $x = 2$.

9. Use the Chain Rule, in Leibniz notation, to find $\dfrac{dy}{dx}$ at the indicated value of $x$.

   a. $y = 3u^2 - 5u + 2, u = x^2 - 1, x = 2$
   b. $y = u^3 - 5(u^3 - 7u)^2, u = \sqrt{x}, x = 4$
   c. $y = 2u^3 + 3u^2, u = x + x^{\frac{1}{2}}, x = 1$
   d. $y = u(u^2 + 3)^3, u = (x + 3)^2, x = -2$
   e. $y = \dfrac{u^3}{u + 1}, u = (x^2 + 1)^3, x = 1$
   f. $y = \dfrac{1}{(1 + u^2)^2}, u = \sqrt{x} - 1, x = 4$
   g. $y = u^5 + u^3, u = \dfrac{3}{v} - 4v, v = 3 - x^2, x = 2$

10. Let $y = f(x^2 + 3x - 5)$. Find $\dfrac{dy}{dx}$ when $x = 1$, given that $f'(-1) = 2$.

Application

**11.** Let $y = g(h(x))$ where $h(x) = \dfrac{x^2}{x + 2}$. If $g'\left(\dfrac{9}{5}\right) = -2$, find $\dfrac{dy}{dx}$ when $x = 3$.

Application

**12.** Find $h'(2)$ given that $h(x) = f(g(x))$, $f(u) = u^2 - 1$, $g(2) = 3$, and $g'(2) = -1$.

## Part C

Application

**13.** a. Write an expression for $h'(x)$ if $h(x) = p(x)q(x)r(x)$.

b. If $h(x) = x(2x + 7)^4(x - 1)^2$, find $h'(-3)$.

14. Show that the tangent to the curve $y = (x^2 + x - 2)^3 + 3$ at the point $(1, 3)$ is also the tangent to the curve at another point.

Thinking/Inquiry/ Problem Solving

**15.** Differentiate $y = \dfrac{x^2(1 - x)^3}{(1 + x)^3}$.

Application **16.** Use mathematical induction to prove that if $u$ is a function of $x$ and $n$ is a positive integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$.

17. If $f(x) = ax + b$ and $g(x) = cx + d$, find the condition (involving $a$, $b$, $c$, and $d$) such that $f(g(x)) = g(f(x))$.

## Technology Extension

Numerical derivatives can be approximated on a TI-83 Plus using **nDeriv(**. To approximate $f'(0)$ for $f(x) = \frac{2x}{x^2 + 1}$:

technology

Press MATH and scroll down to **8:nDeriv(** under the MATH menu.

Press ENTER and the display on the screen will be **nDeriv(**.

To find the derivative, key in the *expression*, the *variable*, the *value* at which we want the derivative, and a value for ∈.

For this example, the display will be **nDeriv(** 2x/(x$^2$ + 1), x, 0, .01).

Press ENTER and the value **1.9998002** will be returned.

Therefore, $f'(0)$ is approximately 1.999 800 02.

A better approximation can be found by using a smaller value for ∈, for example, ∈ = 0.0001. The default value for ∈ is 0.001.

a. Use the **nDeriv(** function on a graphing calculator to find the value of the derivative of each of the following functions at the given point.

b. Determine the actual value of the derivative at the given point using the Rules of Differentiation.

i) $f(x) = x^2$ at $x = 3$

ii) $f(x) = x^3$ at $x = -1$

iii) $f(x) = x^4$ at $x = 2$

iv) $f(x) = x^3 - 6x$ at $x = -2$

v) $f(x) = \sqrt{25 - x^2}$ at $x = 3$

vi) $f(x) = (x^2 + 1)(2x - 1)^4$ at $x = 0$

vii) $f(x) = x^2 + \frac{16}{x} - 4\sqrt{x}$ at $x = 4$

viii) $f(x) = \frac{x^2 - 1}{x^2 + x - 2}$ at $x = -1$

The TI-89 and TI-92 can find exact symbolic and numerical deriviatives. If you have access to either model, try some of the above questions and compare your answers to those found using a TI-83 Plus. Press DIFFERENTIATE under the CALCULATE menu, key d(2x/(x$^2$ + 1), x) | x = 0 and press ENTER.

# Key Concepts Review

Now that you have completed your study of derivatives in Chapter 4, you should be familiar with such concepts as derivatives of polynomial functions, the Product Rule, the Quotient Rule, the Power Rule for Rational Exponents, and the Chain Rule. Consider the following summary to confirm your understanding of key concepts.

- The derivative of $f$ at $a$ is given by $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ or, alternatively,

  by $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.
- The derivative function of $f(x)$ with respect to $x$ is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
- The normal at point $P$ is the line that is perpendicular to the tangent at point $P$.
- For two functions $f$ and $g$, the composite function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.

## Summary of Differentiation Techniques

| Rule | Function Notation | Leibniz Notation |
|---|---|---|
| Constant | $f(x) = k$, $f'(x) = 0$ | $\frac{d}{dx}(k) = 0$ |
| Linear | $f(x) = x$, $f'(x) = 1$ | $\frac{d}{dx}(x) = 1$ |
| Constant Multiple | $f(x) = kg(x)$, $f'(x) = kg'(x)$ | $\frac{d}{dx}(ky) = k\frac{dy}{dx}$ |
| Sum or Difference | $f(x) = p(x) \pm q(x)$,<br>$f'(x) = p'(x) \pm q'(x)$ | $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ |
| Product | $h(x) = f(x)g(x)$<br>$h'(x) = f'(x)g(x) + f(x)g'(x)$ | $\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$ |
| Quotient | $h(x) = \frac{f(x)}{g(x)}$<br>$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ | $\frac{d}{dx}\left[\frac{f(x)}{(x)}\right] = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}$ |
| Chain | $y = f(g(x)), \frac{dy}{dx} = f'(g(x))g'(x)$ | $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, where $u$ is a function of $x$. |
| General Power | $y = [f(x)]^n$, $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$ | $y = u^n$, $\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$,<br>where $u$ is a function of $x$. |

## Career Link wrap-up investigate and apply

### CHAPTER 4: THE ELASTICITY OF DEMAND

An electronics retailing chain has established the monthly price ($p$) − demand ($n_d$) relationship for a Nintendo™ game as

$$n_d(p) = 1000 - 10\frac{(p-1)^{\frac{4}{3}}}{\sqrt[3]{p}}.$$

They are trying to set a price level that will provide maximum revenue ($R$). They know that when demand is *elastic* ($E > 1$), a drop in price will result in higher overall revenues ($R = n_d p$) and that when demand is *inelastic* ($E < 1$), an increase in price will result in higher overall revenues. To complete the questions in this task, you will have to use the elasticity definition of

$$E = -\frac{\frac{\Delta n_d}{n_d}}{\frac{\Delta p}{p}}$$

converted into differential $\left(\frac{\Delta n}{\Delta p} = \frac{dn}{dp}\right)$ notation.

a. Determine the elasticity of demand at \$20 and \$80, classifying these price points as having elastic or inelastic demand. What does this say about where the optimum price is in terms of generating the maximum revenue? Explain. Also calculate the revenue at the \$20 and \$80 price points.

b. Approximate the demand curve as a linear function (tangent) at a price point of \$50. Plot the demand function and its linear approximation on the graphing calculator. What do you notice? Explain this by looking at the demand function.

c. Use your linear approximation to determine the price point that will generate the maximum revenue. (*Hint:* Think about the specific value of $E$ where you won't want to increase or decrease the price to generate higher revenues.) What revenue is generated at this price level?

d. A second game has a price–demand relationship of

$$n_d(p) = \frac{12\,500}{p - 25}.$$

The price is currently set at \$50. Should the company increase or decrease the price? Explain.

# Review Exercise

1. Describe the process of finding a derivative using the definition for $f'(x)$.

2. Use the definition of the derivative to find $f'(x)$ of each of the following functions:

   i) $y = 2x^2 - 5x$   ii) $y = \sqrt{x - 6}$   iii) $y = \dfrac{x}{4 - x}$

3. Differentiate each of the following:

   a. $y = x^2 - 5x + 4$   b. $y = 8 - x^3$   c. $f(x) = x^{\frac{3}{4}}$

   d. $y = -5x^{-4}$   e. $y = \dfrac{7}{3x^4}$   f. $y = \dfrac{1}{x - 3}$

   g. $y = \dfrac{1}{x^2 + 5}$   h. $y = \dfrac{3}{(3 - x^2)^2}$   i. $y = \sqrt{2 - x}$

   j. $y = \sqrt{7x^2 + 4x + 1}$   k. $y = (5x^4 + \pi)^3$   l. $y = x^{-4} + \left(x^3 - 4\right)^{-\frac{2}{5}}$

4. Find the derivative of the given function. In some cases, it will save time if you rearrange the function before differentiating.

   a. $f(x) = \dfrac{2x^3 - 1}{x^2}$   b. $g(x) = \sqrt{x}(x^3 - x)$

   c. $h(x) = \dfrac{8}{3x\sqrt{x}}$   d. $y = \dfrac{3 - \frac{1}{x}}{\phantom{x}}$

   e. $y = \dfrac{x}{3x - 5}$   f. $y = \sqrt{x - 1}(x + 1)$

   g. $f(x) = (\sqrt{x} + 2)^{-\frac{2}{3}}$   h. $y = \sqrt{(x + 3)(x - 3)}$

   i. $y = \dfrac{x^2 + 5x + 4}{x + 4}$   j. $y = \dfrac{x^3 - 27}{x - 3}$

5. Find the derivative, and give your answer in a simplified form.

   a. $y = x^4(2x - 5)^6$   b. $y = x\sqrt{x^2 + 1}$

   c. $y = \dfrac{(2x - 5)^4}{(x + 1)^3}$   d. $y = \dfrac{x}{\phantom{x}}$

   e. $y = \left(\dfrac{10x - 1}{3x + 5}\right)^6$   f. $y = \dfrac{(x^2 - 1)^3}{(x^2 + 1)^3}$

   g. $y = \dfrac{x}{\phantom{x}}$   h. $y = (x - 2)^3(x^2 + 9)^4$

   i. $y = (1 - x^2)^3(6 + 2x)^{-3}$   j. $y = (3x^2 - 2)^2\sqrt{x^2 - 5}$

6. If $f$ is a differentiable function, find an expression for the derivative of each of the following functions:

a. $g(x) = f(x^2)$

b. $h(x) = 2xf(x)$

7. a. If $y = 5u^2 + 3u - 1$ and $u = \frac{18}{x^2 + 5}$, find $\frac{dy}{dx}$ when $x = 2$.

b. If $y = \frac{u + 4}{u - 4}$ and $u = \frac{\sqrt{x} + x}{10}$, find $\frac{dy}{dx}$ when $x = 4$.

c. If $y = f(\sqrt{x^2 + 9})$ and $f'(5) = -2$, find $\frac{dy}{dx}$ when $x = 4$.

8. Find the slope of the line tangent to the graph of $f(x) = (9 - x^2)^{\frac{2}{3}}$ at point $(1, 4)$.

technology

9. For what values of $x$ does the curve $y = -x^3 + 6x^2$ have a slope of $-12$? Of $-15$? Use a graphing calculator to graph the function and confirm your results.

technology

10. a. Find the values of $x$ where the given graph has a horizontal tangent.

i) $y = (x^2 - 4)^5$

ii) $y = (x^3 - x)^2$

b. Use a graphing calculator to graph the function and its tangent at the point to confirm your result.

11. Find the equation of the tangent to each function at the point given.

a. $y = (x^2 + 5x + 2)^4$ at $(0, 16)$

b. $y = (3x^{-2} - 2x^3)^5$ at $(1, 1)$

12. A tangent to the parabola $y = 3x^2 - 7x + 5$ is perpendicular to $x + 5y - 10 = 0$. Determine the equation of the tangent.

13. The line $y = 8x + b$ is tangent to the curve $y = 2x^2$. Find the point of tangency and the value of $b$.

technology

14. a. Using a graphing calculator, graph the function $f(x) = \frac{x^3}{x^2 - 6}$.

b. Using the DRAW function or an equivalent function on your calculator or graphing software, find the equations of the tangents where the slope is zero.

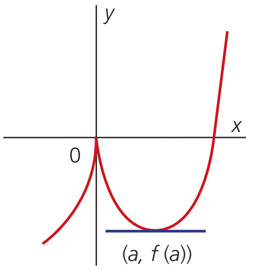
c. Setting $f'(x) = 0$, find the coordinates of the points where the slope is zero.

d. Find the slope of the tangent to the graph at $(2, -4)$. Use the graph to verify that your answer is reasonable.

15. Consider the function $f(x) = 2x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$.

    a. Find the slope of the tangent at the point where the graph crosses the $x$-axis.

    b. Find the value of $a$.




16. A rested student is able to memorize $M$ words after $t$ minutes, where $M = 0.1t^2 - 0.001t^3$.

    a. How many words are memorized in the first 10 min? The first 15 min?

    b. What is the memory rate at $t = 10$? At $t = 15$?

17. A grocery store determines that after $t$ hours on the job, a new cashier can ring up $N(t) = 20 - \frac{30}{\phantom{t^2+1}}$ items per minute.

    a. Find $N'(t)$, the rate at which the cashier's productivity is changing.

    b. According to this model, does the cashier ever stop improving? Why?

18. An athletic-equipment supplier experiences weekly costs of $C(x) = \frac{1}{3}x^3 + 40x + 700$ in producing $x$ baseball gloves per week.

    a. Find the marginal cost, $C'(x)$.

    b. Find the production level $x$, at which the marginal cost is \$76 per glove.

19. A manufacturer of kitchen appliances experiences revenues from the sale of $x$ refrigerators per month of $R(x) = 750x - \frac{x^2}{6} - \frac{2}{3}x^3$ dollars.

    a. Find the marginal revenue, $R'(x)$.

    b. Find the marginal revenue when 10 refrigerators per month are sold.

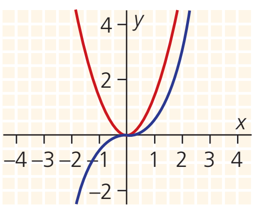
20. An economist has found that the demand function for a particular new product is given by $D(p) = \frac{20}{\phantom{p-1}}$, $p > 1$. Find the slope of the demand curve at the point (5, 10).

# Chapter 4 Test

| Achievement Category | Questions |
|---|---|
| Knowledge/Understanding | 3–7 |
| Thinking/Inquiry/Problem Solving | 11 |
| Communication | 1, 2 |
| Application | 8–10 |

1. Explain when you need to use the Chain Rule.

2. The following graph shows the graphs of a function and its derivative function. Label the graphs $f$ and $f'$ and write a short paragraph stating the criteria you used in making your selection.




3. Use the definition of the derivative to find $\frac{d}{dx}(x - x^2)$.

4. Find $\frac{dy}{dx}$ for each of the following:

   a. $y = \frac{1}{3}x^3 - 3x^{-5} + 4\pi$

   b. $y = 6(2x - 9)^5$

   c. $y = \frac{2}{\phantom{x}} + \frac{x}{\sqrt{3}} + 6\sqrt[3]{x}$

   d. $y = \left(\frac{x^2 + 6}{3x + 4}\right)^5$ Leave your answer in a simplified factored form.

   e. $y = x^2\sqrt[3]{6x^2 - 7}$ Simplify your answer.

   f. $y = \frac{4x^5 - 5x^4 + 6x - 2}{x^4}$ Simplify your answer.

5. Find the slope of the tangent to the graph of $y = (x^2 + 3x - 2)(7 - 3x)$ at $(1, 8)$.

6. Find $\frac{dy}{dx}$ at $x = -2$ for $y = 3u^2 + 2u$ and $u = \sqrt{x^2 + 5}$.

7. Find the equation of the tangent to $y = (3x^{-2} - 2x^3)^5$ at point $(1, 1)$.

8. The amount of pollution in a certain lake is $P(t) = \left(t^{\frac{1}{4}} + 3\right)^3$, where $t$ is measured in years, and $P$ is measured in parts per million (p.p.m.). At what rate is the amount of pollution changing after 16 years?

9. At what point on the curve $y = x^4$ does the normal have a slope of 16?

10. Find the points on the curve $y = x^3 - x^2 - x + 1$ where the slope of the tangent is horizontal.

11. For what values of $a$ and $b$ will the parabola $y = x^2 + ax + b$ be tangent to the curve $y = x^3$ at point $(1, 1)$?

# Cumulative Review

## CHAPTERS 1–4

1. Sketch the graph of $y = (x - 4)(x - 1)(x + 3)$.

2. Sketch the graph of $y = -(x + 1)^2(x - 2)$.

3. Find the polynomial function whose graph passes through the following points: $(1, -25)$, $(2, -20)$, $(3, 3)$, $(4, 56)$, $(5, 151)$.

4. a. Divide $x^3 - 3x^2 + 4$ by $x - 2$.
   b. Divide $3x^3 - 4x^2 + 11x - 2$ by $x + 3$.
   c. Divide $x^4 - 5x^2 + x - 1$ by $x^2 - x + 1$.

5. Find the remainder when $2x^3 - x^2 + 7x + 1$ is divided by $x - 2$.

6. When $3x^5 - 6x^3 + kx + 2$ is divided by $x - 1$, the remainder is 5. Find the value of $k$.

7. If $x - 3$ is a factor of $x^3 + kx^2 - 4x + 12$, where $k \in R$, find the value of $k$.

8. Determine whether or not $x - 2$ is a factor of $x^4 - 2x^3 + 5x^2 - 6x - 8$.

9. One factor of $x^3 - 2x^2 - 5x + 6$ is $x + 2$. Determine the other factors.

10. Factor fully.
    a. $x^3 + 3x^2 - 18x - 40$
    b. $x^3 + 5x^2 - 4x - 20$
    c. $2x^3 + x^2 - 8x - 4$
    d. $5x^3 + 8x^2 + 21x - 10$

11. Solve for $x$ ($x \in C$).
    a. $x^3 + 3x^2 - 4 = 0$
    b. $x^4 + 5x^2 - 36 = 0$
    c. $x^3 + 4x^2 + x - 6 = 0$
    d. $2x^3 - x^2 - 2x + 1 = 0$
    e. $x^3 + x^2 - 5x + 3 = 0$
    f. $3x^3 - 4x^2 + 4x - 1 = 0$

12. Find the sum and product of the roots of $2x^2 + 8x + 5 = 0$.

13. Find the quadratic equation whose roots are the squares of the roots of $x^2 - 9x + 2 = 0$.

14. Solve each of the following, $x \in R$:

a. $x^2 - x - 6 < 0$

b. $(x + 2)(x - 1)(x - 3) \geq 0$

15. Solve for $x$.

a. $|x - 2| < 5$

b. $|2x - 3| \leq 5$

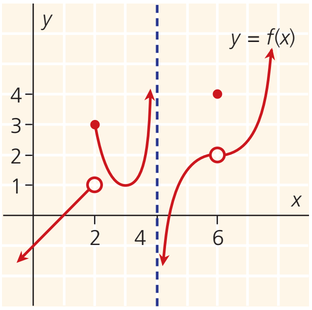
c. $|3x + 1| > 16$

16. The displacement (in metres) of an object is given by $s(t) = 2t^2 + 3t + 1$, where $t$ is the time in seconds.

a. Find the average velocity from $t = 1$ to $t = 4$.

b. Find the instantaneous velocity at $t = 3$.

17. A cylinder has a volume of $200\pi$ cm$^3$. Its height is 3 cm greater than its radius. Determine the radius of the cylinder. (The volume of a cylinder is $V = \pi r^2 h$.)




18. a. Determine $f(2)$.

b. Determine $\lim_{x \to 2^-} f(x)$.

c. Determine $\lim_{x \to 2^+} f(x)$.

d. Determine $\lim_{x \to 6} f(x)$.

e. Does $\lim_{x \to 4} f(x)$ exist? Justify your answer.

19. Sketch a function $f(x)$ that satisfies the following conditions:

- $f(x)$ is increasing;
- $\lim_{x \to 2^-} f(x) = 5$;
- $\lim_{x \to 2^+} f(x) = 8$.

20.
$$f(x) = \begin{cases} x^2 + 1, x < 2 \\ 2x + 1, x = 2 \\ -x + 5, x > 2 \end{cases}$$

Determine where $f(x)$ is discontinuous and justify your answer.

21. Use your graphing calculator to estimate $\lim_{x \to 1} \frac{x^4 + 2x^3 - 2x^2 + 2x - 3}{x^2 - 1}$.

22. If $\lim_{h \to 0} \frac{(4 + h)^3 - 64}{h}$ is the slope of the tangent to $y = f(x)$ at $x = 4$, what is $f(x)$?

23. Use algebraic methods to evaluate each of the following (if they exist):

a. $\lim\limits_{x \to 0} \dfrac{2x^2 + 1}{x - 5}$ b. $\lim\limits_{x \to 5} \dfrac{1}{x - 5}$ c. $\lim\limits_{x \to -3} \dfrac{\frac{1}{x} + \frac{1}{3}}{x + 3}$

d. $\lim\limits_{x \to 0} \dfrac{x^3 + 4x^2 + 2x}{3x^5 + 2x^3 + x}$ e. $\lim\limits_{x \to 2} \dfrac{x - 2}{x^3 - 8}$ f. $\lim\limits_{x \to 3} \dfrac{\sqrt{x + 1} - 2}{}$

24. Use the definition of derivative to find $f'(x)$ if

a. $f(x) = 3x^2 + x + 1.$ b. $f(x) = \frac{1}{x}.$

25. Use any of the rules you have learned to find the derivatives of the following functions:

a. $y = x^3 - 4x^2 + 5x + 2$ b. $y = \sqrt{2x^3 + 1}$

c. $y = \dfrac{2x}{x + 3}$ d. $y = (x^2 + 3)^2 (4x^5 + 5x + 1)$

e. $y = \dfrac{(4x^2 + 1)^5}{(3x - 2)^3}$ f. $y = [x^2 + (2x + 1)^3]^5$

26. Find the equation of the tangent to $y = \dfrac{18}{(x + 2)^2}$ at point (1, 2).

27. Find the equation of the tangent to $y = x^2 - 4x + 1$ that is perpendicular to the line with equation $3x + 6y - 2 = 0$.

28. If $y = 6u^3 + 2u^2 + 5u - 2$ and $u = \dfrac{1}{x^3 + 2}$, find $\dfrac{dy}{dx}$.

29. Find the slope of the tangent to $y = x^2 + 9x + 9$ at the point where the curve intersects the line at $y = 3x$.

30. In 1980, the population of Smalltown, Ontario, was 1100. After a time $t$, in years, the population is given by $p(t) = 2t^2 + 6t + 1100$.

a. Find $p'(t)$, the function that describes the rate of change of the population.

b. Find the rate of change of the population in 1990.

c. In what year is the rate of change of the population 94 people per year?